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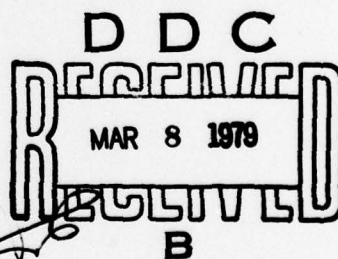
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(6) DEGENERACY AND THE MORE-FOR-LESS PARADOX.

by

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## ABSTRACT

M. Ryan conjectured that in the "more-for-less (nothing)" situation in the distribution model of linear programming a spatial separation of markets would necessarily occur if shipments were increased to the point where the "more-for-less (nothing)" paradox just disappears. In the paper, we prove that Ryan's conjecture is true.

Key Words and Phrases: More-for-Less Paradox, spatial Separation of Markets, Degenerate Transportation Models, Distribution Models.

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### Introduction

The "more-for-less (nothing)" paradox in the distribution (or "transportation") model of linear programming occurs when the supplies  $a_i$ , demands  $b_j$  and unit shipment costs  $c_{ij}$  (all assumed positive) are such that in the problem:

$$(1) \quad \min \sum_{i,j} c_{ij} x_{ij}$$

$$\text{with } \sum_j x_{ij} = a_i, \quad i = 1, \dots, m$$

$$\sum_i x_{ij} = b_j, \quad j = 1, \dots, n$$

$$x_{ij} \geq 0, \quad \sum_i a_i = \sum_j b_j$$

some  $a_i$  (hence also some  $b_j$ ) can be increased without lowering others while the minimal total shipment cost becomes less than (equal to) that in problem (1).

The major elucidation of this paradox thus far is the paper [1] of Charnes and Klingman in which (among other results) they establish

Theorem 1: If (1) has a non-degenerate optimal basic solution, then (1) exhibits the more-for-less paradox iff  $R_{i_0} + K_{j_0} < 0$  for some  $(i_0, j_0)$  where the  $R_i, K_j$  are dual evaluators for this solution.

Although not explicitly stated, their proof also shows, if instead  $R_{i_0} + K_{j_0} = 0$ , that (1) exhibits the "more-for-nothing" paradox.

In [2], M. Ryan, working with examples, conjectured in the situation of Theorem 1 that if one increased shipments maximally to where the paradox

just stops, then there would be a spatial decomposition of this "market," i.e., every basic optimal solution to the new problem (1) would be degenerate.

It is the purpose of this paper to relate the more-for-less situation to degeneracy and to establish that Ryan's conjecture is true.

### A Minimum-minimorum Embedding

All allowable shipment increases in the more-for-less (nothing) situation in (1) can be studied by embedding them in the problem:

$$\begin{aligned}
 (2) \quad & \min \sum_{i,j} c_{ij} x_{ij} \\
 & \text{with } \sum_j x_{ij} \geq a_i, \quad i = 1, \dots, m \\
 & \quad \sum_i x_{ij} \geq b_j, \quad j = 1, \dots, n \\
 & \quad x_{ij} \geq 0
 \end{aligned}$$

Thus, for example, optimal solution of (2) would yield a minimum-minimorum over all relevant shipment increases.

By adding slack variables  $r_i, s_j$  in problem (2) we obtain the equivalent form:

$$\begin{aligned}
 (2A) \quad & \min \sum_{i,j} c_{ij} x_{ij} \\
 & \text{with } \sum_j x_{ij} - r_i = a_i, \quad i = 1, \dots, m \\
 & \quad \sum_i x_{ij} - s_j = b_j, \quad j = 1, \dots, n \\
 & \text{and } x_{ij}, r_i, s_j \geq 0
 \end{aligned}$$

Note that the rank of the coefficient matrix in (2A) is  $m + n$  and that  $\sum_i r_i = \sum_j s_j$ . If we are in the more-for-less (nothing) situation, then an optimal basic solution can have some  $r_i > 0$ , hence some  $s_j > 0$ , hence at most  $m + n - 2$  positive  $x_{ij}$ . By fixing such  $\hat{r}_i, \hat{s}_j$  values in (2A) and moving them to the right-hand side of the equations to retrieve the form (1), the corresponding  $\hat{x}_{ij}$  values comprise a degenerate optimal solution. Thus we have established, Theorem 2: In the more-for-less (nothing) situation of (1) with increased shipments there exist degenerate optimal basic solutions.

### Ryan's Conjecture

We next consider problems (1A)

$$\begin{aligned} \min & \sum_{i,j} c_{ij} x_{ij} \\ \text{with } & \sum_j x_{ij} = a_i + \hat{r}_i, \quad i = 1, \dots, m \\ & \sum_i x_{ij} = b_j + \hat{s}_j, \quad j = 1, \dots, n \end{aligned}$$

where  $\hat{r}_i, \hat{s}_j$  are from optimal basic solutions to problem (2A) with some  $\hat{r}_i, \hat{s}_j > 0$ .

Lemma: Dual evaluators  $R_i, K_j$  for a non-degenerate optimal basic solution to (1A) with maximal  $\hat{r}_i, \hat{s}_j$  satisfy  $R_i + K_j > 0$  for all  $(i, j)$ .

Proof: (i) If some  $R_{i_0} + K_{j_0} < 0$ , then by Theorem 1 we would be in the more-for-less situation, contradicting the minimum-minimorum quality of optimal solutions to (1A).

(ii) If some  $R_{i_0} + K_{j_0} = 0$ , we would be in the more-for-nothing situation, contradicting the maximality of the  $\hat{r}_i, \hat{s}_j$ .

Thus  $R_i + K_j > 0$  for all  $(i, j)$  in a non-degenerate solution of (1A).

Q.E.D.



We next consider an arbitrary non-degenerate basic optimal solution to (1A). By the lemma, the associated  $R_i, K_j$  satisfy  $R_i + K_j > 0$ , all  $(i, j)$ . Choose  $i_0, j_0$  for which  $\hat{r}_{i_0}, \hat{s}_{j_0} > 0$ . Since the basic optimal solution is non-degenerate, there is a "stepping-stone" path for cell  $(i_0, j_0)$  on basic cells all of which contain positive shipment amounts. Thus there is a  $\delta > 0$  for which  $\delta$  is less than the minimum amount on a basic cell in this path.

Thus we reduce by  $\delta$  the supply at row  $i_0$  and the demand at column  $j_0$  without losing our current basis as optimal i.e., the same  $R_i, K_j$  and optimality conditions hold for the reduced problem:

$$(1AA) \quad \min \sum_{i,j} c_{ij} x_{ij}$$

$$\text{with } \sum_j x_{ij} = a_i + \hat{r}_i, \quad i \neq i_0$$

$$\sum_i x_{ij} = b_j + \hat{s}_j, \quad j \neq j_0$$

and

$$\sum_j x_{i_0 j} = a_{i_0} + \hat{r}_{i_0} - \delta$$

$$\sum_i x_{i j_0} = b_{j_0} + \hat{s}_{j_0} - \delta$$

But the optimal value in (1AA) is (\*\*)

$$\sum_i R_i (a_i + \hat{r}_i) + \sum_j K_j (b_j + \hat{s}_j) - \delta (R_{i_0} + K_{j_0})$$

whereas that in (1A) is

$$(*) \quad \sum_i R_i (a_i + \hat{r}_i) + \sum_j K_j (b_j + \hat{s}_j)$$

contradicting the optimality of (\*) for (1A), the equivalent equation form to the minimum-minimorum form (2).

Q. E. D.



Thus we have proven Ryan's conjecture,

Theorem 3: No non-degenerate basic optimal solution to problem (1A) exists,  
with maximal  $\hat{r}_i$ ,  $\hat{s}_j$ .

References

- [1] Charnes, A. and D. Klingman. "The 'More-for-Less' Paradox in the Distribution Model." Cahiers de Centre d'Etudes de Recherche Operationnelle, 13, 1, 1971.
  
- [2] Ryan, M. J. "More on the More-for-Less Paradox in the Distribution Model." Proceedings, International Symposium on Extremal Methods and Systems Analysis on the Occasion of Prof. A. Charnes' 60th Birthday, 1977.

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